Chapter

SPIN AND THE FOURTH DIMENSION

V.V. Varlamov[*]
Siberian State Industrial University, Novokuznetsk, Russia

Abstract

A group-theoretic interpretation of the periodic system of elements is given within the framework of the weight diagram of the Lie algebra $\mathfrak{so}(4,4)$ of the fourth rank, where the four quantum numbers n, l, m, s correspond to the eigenvalues (weights) of the Cartan generators of the maximal Abelian subalgebra (the maximal torus of the group $\mathrm{SO}(4,4)$). It is shown that the root system of the algebra $\mathfrak{so}(4,4)$ forms a regular four-dimensional self-dual polyhedron (24-cell). The action of the fourth Cartan generator associated with spin leads to a splitting of the Cartan-Weyl basis of the algebra $\mathfrak{so}(4,4)$ into two structurally identical bases, each of which is isomorphic to the Yao basis of the subalgebra $\mathfrak{so}(4,2)$ (the Lie algebra of the conformal group). At this point, a four-dimensional 24-cell is projected onto two three-dimensional cuboctahedra, each of which defines the root system of the subalgebra $\mathfrak{so}(4,2)$. This splitting physically corresponds to spin doubling (two-valuedness). The period doubling associated with the sequence of period lengths 2, 8, 8, 18, 18, 32, 32, ... of the periodic system of elements is explained by the action of the fourth Cartan generator.

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[*]E-mail address: varlamov@sibsiu.ru

1. Introduction

The concept of spin was introduced by Pauli in 1925, explaining the doublet structure of the spectrum of alkali metals (the anomalous Zeeman effect): ”The doublet structure of the alkali spectra, as well as the violation of the Larmor theorem are, according to this point of view, a result of a classically not describable two-valuedness of the quantum-theoretical properties of the valence electron” [1]. Later, Van der Waerden noted that this “classically non-describable two-valuedness” of electron we call spin [2, P. 215]. From this it can be seen that the concept of spin arose from the analysis of a spectroscopic problem related to the structure of the periodic system of chemical elements.

However, to this day, 100 years later, spin remains a mystery. All attempts to connect spin with three-dimensional mechanical representations proved unsuccessful. Spin is the fourth degree of freedom, which cannot be described by the classical kinematic model in three dimensions. The concept of spin requires the introduction of a fourth dimension for its adequate description.

The group-theoretic description of the periodic system of elements allows us to explicitly link spin with the fourth dimension. As is known, immediately after D.I. Mendeleev’s discovery of the periodic table of chemical elements, attempts began to be made to mathematically describe (mathematize) the periodic law (for the history of the issue, see [3, 4]). It is not surprising that the group theory turned out to be the most suitable mathematical structure for describing the phenomenon of periodicity, i.e., repeatability (cyclicity). Historically, graphical visualization of the periodic law has been very important. In addition to the initial 2-dimensional tabular representation (the Mendeleev and Meyer tables), there are many other different forms: spiral, helical and pyramidal models of the periodic table, as well as many exotic forms (for more details, see [5, 6, 7]). The main feature of these models is the construction of a system of chemical elements depending on the increase in atomic weight. However, quantum mechanics has led to the understanding that the main structural characteristic of the periodic law is not a linear increase in atomic weight, but a structure determined by the order of quantum numbers. Polygonfläche by G. Haenzel [8] is historically the first model of this kind, where the structure of a periodic system is determined by the order of quantum numbers forming systems of concentric polygons on a plane (see Figure 1). V. Finke in the article [9] presents the Haenzel’s Polygonfläche in three-dimensional space, associating it with the Madelung numbering. On the other hand, in the Rumer-Fet-Barut approach [10, 11, 12], the structure of the ”left-sided” Janet table is associated with the quantum numbers of the conformal group $\mathrm{SO}(4,2)$ and its tensor extension G_F [10, 11], see also [13, 14, 15, 16]. As is known, the Janet table is a two-dimensional graphical representation of the Madelung rule. Thus, by means of Madelung numbering, there is a correspondence between the Haenzel-Finke model and the Rumer-Fet-Barut group-theoretic approach. However, a common disadvantage of Haenzel’s Polygonfläche and Finke’s three-dimensional system is the artificial representation of the fourth quantum number s (spin) as two points on transversals (see Figure 1). In turn, in the group-theoretic approach, the first three quantum numbers n,l, and m correspond to the eigenvalues of Cartan generators forming the maximal Abelian subalgebra of the Lie algebra $\mathfrak{so}(4,2)$ of the conformal group. The algebra $\mathfrak{so}(4,2)$ has the third rank. To include the fourth quantum number s on an equal basis with n, l, m, i.e. as an eigenvalue of the Cartan generator, a transition to the

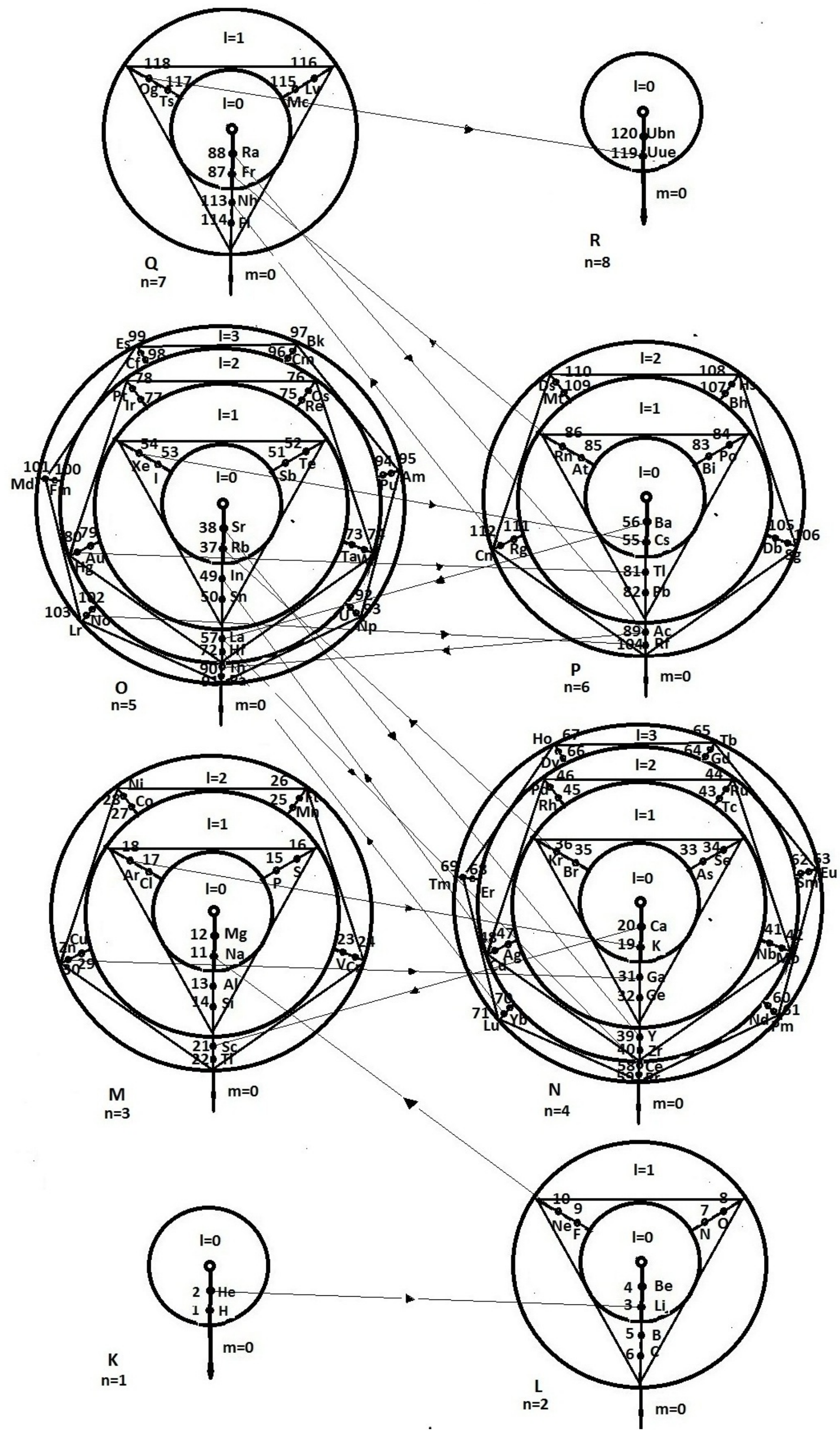

Figure 1. Polygonfläche and the Periodic System.

Lie algebra of the fourth rank is required. It is shown in [17] that such an algebra is $\mathfrak{so}(4,4)$ – the Lie algebra of the group $\mathrm{SO}(4,4)$.

The root diagram of the algebra $\mathfrak{so}(4,4)$ is defined in a four-dimensional weight space, forming a regular four-dimensional self-dual polyhedron (24-cell). The corresponding spin generator L_{78} commutes with all generators of the subalgebra $\mathfrak{so}(4,2)$, which leads to splitting ("doubling") the Cartan-Weyl basis of the algebra $\mathfrak{so}(4,4)$ into two structurally identical bases of the subalgebra $\mathfrak{so}(4,2)$, which geometrically correspond to two three-dimensional orthographic projections (cuboctahedra) of a four-dimensional 24-cell. As a consequence, the four-dimensional weight diagram of the algebra $\mathfrak{so}(4,4)$ can be represented by two three-dimensional projections, each of which is isomorphic to the weight diagram of the subalgebra $\mathfrak{so}(4,2)$ [17]. It should be noted that the group $\mathrm{SO}(4,4)$ has a higher symmetry than the tensor extensions G_F and G_O of the Ostrovsky-Fet scheme: the Lie algebra $\mathfrak{so}(4,4)$ contains 28 independent generators, while the Lie algebra $\mathfrak{so}(4,2)\otimes\mathfrak{su}(2)\otimes\mathfrak{su}(2)'$ of the groups G_F and G_O has 21 independent generators. The main disadvantage of the Ostrovsky-Fet scheme of the group-theoretic description of period doubling is the artificial nature of the introduction of the fifth quantum number, which has no real analogue, since all states (elements) of the periodic system are described by the four quantum numbers $(n,l,m.s)$. Moreover, the introduction of the fifth quantum number s' leads to a change in the Madelung rule (the Fet rule [11]). It is shown that the fourth Cartan generator L_{78} (spin generator), due to the higher symmetry of the group $\mathrm{SO}(4,4)$, combines ("absorbs") the functions of the Cartan generators $\boldsymbol{\tau}_3$ and $\boldsymbol{\tau}'_3$ in Ostrovsky-Fet scheme, which avoids the introduction of an additional (non-physical) quantum number s'. In paragraph 2 of this article, two three-dimensional projections of the weight diagram of the algebra $\mathfrak{so}(4,4)$ are combined into a double $\mathrm{SO}(4,2)$-tower, each node of which (the finite-dimensional representation of the group $\mathrm{SO}(4,2)$) is associated with the corresponding element of the periodic table according to the Madelung numbering. The sweep of the double $\mathrm{SO}(4,2)$-tower onto the Janet table and the correspondence with the Rumer-Fet-Barut model is established through the Madelung basis.

2. The Periodic Table and the Group $\mathrm{SO}(4,4)$

The four quantum numbers n, l, m, s define four degrees of freedom. The first three quantum numbers n, l, m have a clear geometric interpretation in Haenzel's Polygonfläche [8] and the three-dimensional Finke system [9], see also [17]. However, the fourth quantum number s (spin) goes beyond the three-dimensional representation and requires the introduction of a fourth dimension for its adequate description[1].

The theory of groups allows us to connect spin with the fourth dimension. Thus, according to the group-theoretic description of the periodic table [11, 12, 13], the first three quantum numbers n, l, and m correspond to the eigenvalues ν, λ, and μ_λ of the generators L_{56}, L_{12} and L_{34} forming the Cartan subalgebra $\mathfrak{K}$ of the Lie algebra $\mathfrak{so}(4,2)$ of the conformal group $\mathrm{SO}(4,2)$. $\mathfrak{so}(4,2)$ is a Lie algebra of the third rank, therefore all root and

[1] In the Haenzel and Finke systems, spin is artificially accounted for as points on the transversals. This clearly shows that the concept of spin does not fit into three-dimensional space, and all three-dimensional mechanical interpretations (like the Goudsmit-Uhlenbeck spinning top) are unable to describe spin by definition.

weight diagrams of this algebra are three-dimensional systems. An adequate description of spin in the framework of a group-theoretic scheme requires a transition to a fourth-rank Lie algebra. Such an algebra is $\mathfrak{so}(4,4)$ – the Lie algebra of the rotation group $\mathrm{SO}(4,4)$ of the eight-dimensional pseudo-Euclidean space $\mathbb{R}^{4,4}$.

A special pseudo-orthogonal group in eight dimensions, $\mathrm{SO}(4,4)$, corresponds to the rotation group of the eight-dimensional pseudo-Euclidean space $\mathbb{R}^{4,4}$, or, equivalently, the set of 8×8 orthogonal matrices leaving a quadratic form

$$Q(\boldsymbol{r}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2 = \boldsymbol{r}^T\boldsymbol{r},$$

where $\boldsymbol{r} = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]^T$, invariant.

The structure of the corresponding Lie algebra $\mathfrak{so}(4,4)$ is determined by the commutation properties of its generators $\mathsf{L}_{\alpha\beta}$. $\mathsf{L}_{\alpha\beta}$ form the basis of the algebra $\mathfrak{so}(4,4)$. The number of independent generators is easy to find: out of 64 possible combinations of the indices α and β, eight combinations disappear by virtue of $\mathsf{L}_{\alpha\alpha} = 0$, this reduces the number of generators to 56. Moreover, by virtue of $\mathsf{L}_{\alpha\beta} = -\mathsf{L}_{\beta\alpha}$ only 28 independent generators remain, the number of which can also be obtained using the formula $n(n-1)/2$, where $n = p + q$ is the dimension of the space $\mathbb{R}^{p,q}$. Thus,

$$\mathsf{L} \Leftrightarrow \begin{bmatrix} 0 & \mathsf{L}_{12} & \mathsf{L}_{13} & \mathsf{L}_{14} & \mathsf{L}_{15} & \mathsf{L}_{16} & \mathsf{L}_{17} & \mathsf{L}_{18} \\ & 0 & \mathsf{L}_{23} & \mathsf{L}_{24} & \mathsf{L}_{25} & \mathsf{L}_{26} & \mathsf{L}_{27} & \mathsf{L}_{28} \\ & & 0 & \mathsf{L}_{34} & \mathsf{L}_{35} & \mathsf{L}_{36} & \mathsf{L}_{37} & \mathsf{L}_{38} \\ & & & 0 & \mathsf{L}_{45} & \mathsf{L}_{46} & \mathsf{L}_{47} & \mathsf{L}_{48} \\ & & & & 0 & \mathsf{L}_{56} & \mathsf{L}_{57} & \mathsf{L}_{58} \\ & & & & & 0 & \mathsf{L}_{67} & \mathsf{L}_{68} \\ & & & & & & 0 & \mathsf{L}_{78} \\ & & & & & & & 0 \end{bmatrix}. \tag{1}$$

The system of 28 generators $\mathsf{L}_{\alpha\beta}$ of the algebra $\mathfrak{so}(4,4)$ satisfies the following permutation relations:

$$[\mathsf{L}_{\alpha\beta}, \mathsf{L}_{\gamma\delta}] = i\left(g_{\alpha\delta}\mathsf{L}_{\beta\gamma} + g_{\beta\gamma}\mathsf{L}_{\alpha\delta} - g_{\alpha\gamma}\mathsf{L}_{\beta\delta} - g_{\beta\delta}\mathsf{L}_{\alpha\gamma}\right), \tag{2}$$

where $\alpha, \beta, \gamma, \delta = 1, \ldots, 8$, $\alpha \neq \beta$, $\gamma \neq \delta$, while $g_{11} = g_{22} = g_{33} = g_{44} = 1$, $g_{55} = g_{66} = g_{77} = g_{88} = -1$. Thus, we have a 28-dimensional Lie algebra $\mathfrak{so}(4,4)$.

Let's find the maximal subset of commuting generators of the algebra $\mathfrak{so}(4,4)$. As is known, two generators commute if they do not have common indexes. It is easy to see that among the generators of the algebra $\mathfrak{so}(4,4)$, four generators satisfy this condition:

$$\mathsf{L}_{12},\ \mathsf{L}_{34},\ \mathsf{L}_{56},\ \mathsf{L}_{78}. \tag{3}$$

The four generators $\{\mathsf{L}_{12}, \mathsf{L}_{34}, \mathsf{L}_{56}, \mathsf{L}_{78}\}$ form the basis of *maximal abelian subalgebra* $\mathfrak{K} \subset \mathfrak{so}(4,4)$ (*Cartan subalgebra*). The generators (3) are called *Cartan generators*. The dimension of the subalgebra $\mathfrak{K}$ defines the *rank* of the Lie algebra, hence $\mathfrak{so}(4,4)$ is a Lie algebra of the fourth rank. Thus, all root and weight diagrams for $\mathfrak{so}(4,4)$ will be four-dimensional.

The *Cartan-Weyl basis* of the algebra $\mathfrak{so}(4,4)$ contains 28 generators,

$$\left\{\mathsf{L}_{12},\mathsf{L}_{34},\mathsf{L}_{56},\mathsf{L}_{78},{}^1\mathsf{K}_+,{}^1\mathsf{K}_-,{}^1\mathsf{J}_+,{}^1\mathsf{J}_-,{}^1\mathsf{T}_+,{}^1\mathsf{T}_-,{}^1\mathsf{S}_+,{}^1\mathsf{S}_-,{}^1\mathsf{P}_+,{}^1\mathsf{P}_-,{}^1\mathsf{Q}_+,{}^1\mathsf{Q}_-,\right.$$
$$\left.{}^2\mathsf{K}_+,{}^2\mathsf{K}_-,{}^2\mathsf{J}_+,{}^2\mathsf{J}_-,{}^2\mathsf{T}_+,{}^2\mathsf{T}_-,{}^2\mathsf{S}_+,{}^2\mathsf{S}_-,{}^2\mathsf{P}_+,{}^2\mathsf{P}_-,{}^2\mathsf{Q}_+,{}^2\mathsf{Q}_-\right\}, \quad (4)$$

including 4 Cartan generators and 24 Weyl generators (for more details, see [17]. Thus, the root diagram of the algebra $\mathfrak{so}(4,4)$ is defined in a four-dimensional weight space whose coordinate axes are the Cartan generator axes, and the 24 Weyl generator axes form a regular polyhedron in four-dimensional space. Figure 2 shows the root diagram of $\mathfrak{so}(4,4)$ as a 24-cell[2].

The fourth generator L_{78}, understood as a spin generator[3], commutes with all 15 generators of the subalgebra $\mathfrak{so}(4,2)$. As a consequence, the Cartan-Weyl basis (4) splits into two structurally identical bases

$$\left\{{}^1\mathsf{K}_3,{}^1\mathsf{J}_3,{}^1\mathsf{T}_0,{}^1\mathsf{S}_0,{}^1\mathsf{P}_0,{}^1\mathsf{Q}_0,{}^1\mathsf{K}_+,{}^1\mathsf{K}_-,\right.$$
$$\left.{}^1\mathsf{J}_+,{}^1\mathsf{J}_-,{}^1\mathsf{T}_+,{}^1\mathsf{T}_-,{}^1\mathsf{S}_+,{}^1\mathsf{S}_-,{}^1\mathsf{P}_+,{}^1\mathsf{P}_-,{}^1\mathsf{Q}_+,{}^1\mathsf{Q}_-\right\}, \quad (5)$$

$$\left\{{}^2\mathsf{K}_3,{}^2\mathsf{J}_3,{}^2\mathsf{T}_0,{}^2\mathsf{S}_0,{}^2\mathsf{P}_0,{}^2\mathsf{Q}_0,{}^2\mathsf{K}_+,{}^2\mathsf{K}_-,\right.$$
$$\left.{}^2\mathsf{J}_+,{}^2\mathsf{J}_-,{}^2\mathsf{T}_+,{}^2\mathsf{T}_-,{}^2\mathsf{S}_+,{}^2\mathsf{S}_-,{}^2\mathsf{P}_+,{}^2\mathsf{P}_-,{}^2\mathsf{Q}_+,{}^2\mathsf{Q}_-\right\}, \quad (6)$$

each of which is isomorphic to the Yao basis [19] for the group algebra of the twofold covering $\mathbf{Spin}_+(4,2)\simeq \mathrm{SU}(2,2)$. The bases (5) and (6) define two root systems

$$\left.\begin{array}{lll}
\boldsymbol{\alpha}({}^1\mathsf{K}_+)=(+1,+1,0), & \boldsymbol{\alpha}({}^1\mathsf{T}_+)=(+1,0,+1), & \boldsymbol{\alpha}({}^1\mathsf{P}_+)=(0,+1,+1),\\
\boldsymbol{\alpha}({}^1\mathsf{K}_-)=(-1,-1,0), & \boldsymbol{\alpha}({}^1\mathsf{T}_-)=(-1,0,-1), & \boldsymbol{\alpha}({}^1\mathsf{P}_-)=(0,-1,-1),\\
\boldsymbol{\alpha}({}^1\mathsf{J}_+)=(-1,+1,0), & \boldsymbol{\alpha}({}^1\mathsf{S}_+)=(-1,0,+1), & \boldsymbol{\alpha}({}^1\mathsf{Q}_+)=(0,-1,+1),\\
\boldsymbol{\alpha}({}^1\mathsf{J}_-)=(+1,-1,0), & \boldsymbol{\alpha}({}^1\mathsf{S}_-)=(+1,0,-1), & \boldsymbol{\alpha}({}^1\mathsf{Q}_-)=(0,+1,-1),
\end{array}\right\} \quad (7)$$

$$\left.\begin{array}{lll}
\boldsymbol{\alpha}({}^2\mathsf{K}_+)=(+1,-1,0), & \boldsymbol{\alpha}({}^2\mathsf{T}_+)=(+1,0,-1), & \boldsymbol{\alpha}({}^2\mathsf{P}_+)=(0,+1,-1),\\
\boldsymbol{\alpha}({}^2\mathsf{K}_-)=(-1,+1,0), & \boldsymbol{\alpha}({}^2\mathsf{T}_-)=(-1,0,+1), & \boldsymbol{\alpha}({}^2\mathsf{P}_-)=(0,-1,+1),\\
\boldsymbol{\alpha}({}^2\mathsf{J}_+)=(+1,+1,0), & \boldsymbol{\alpha}({}^2\mathsf{S}_+)=(+1,0,+1), & \boldsymbol{\alpha}({}^2\mathsf{Q}_+)=(0,+1,+1),\\
\boldsymbol{\alpha}({}^2\mathsf{J}_-)=(-1,-1,0), & \boldsymbol{\alpha}({}^2\mathsf{S}_-)=(-1,0,-1), & \boldsymbol{\alpha}({}^2\mathsf{Q}_-)=(0,-1,-1),
\end{array}\right\} \quad (8)$$

$$\boldsymbol{\alpha}(\mathsf{L}_3)=(0,0,0),\quad \boldsymbol{\alpha}(\mathsf{A}_3)=(0,0,0),\ \boldsymbol{\alpha}(\Delta_3)=(0,0,0).$$

Graphically, the root systems (7) and (8) can be represented by two cuboctahedra shown in Figure 3.

[2] A 24-cell or *octaplex* (other names: polyoctahedron, icositetrachoron) is the convex regular self-dual polyhedron (polytope) in four-dimensional space. The octaplex contains 24 vertices, 96 edges, 96 triangular faces, and 24 octahedral cells with six meeting at each vertex, and three at each edge. The symmetry group F_4 (Coxeter group [18]) of this polyhedron has the order 1152. An octaplex is the only self-dual regular polytope of dimension greater than 2 that is not a simplex. This is the reason for the uniqueness of the octaplex: unlike the other five regular four-dimensional polyhedra (pentachoron, tesseract, hexadecachoron, hypericosahedron,

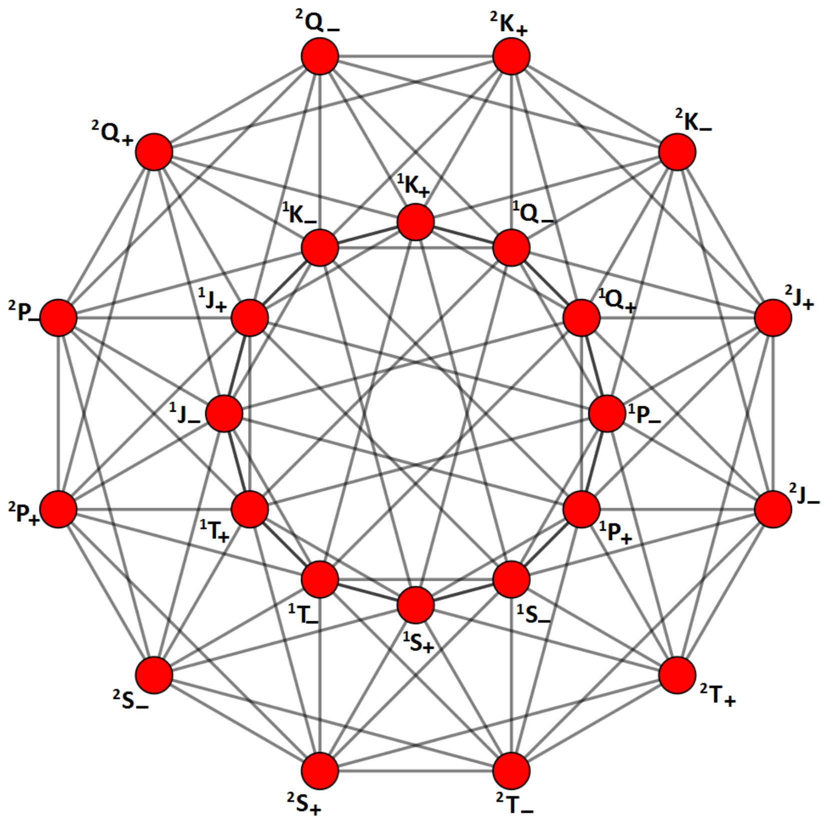

Figure 2. Root diagram of the Lie algebra $\mathfrak{so}(4,4)$. Weyl generators form the vertices of a 24-cell in four-dimensional space.

Figure 4 shows two three-dimensional projections ($\mathrm{SO}(4,2)$-towers) of the weight diagram of the Lie algebra $\mathfrak{so}(4,4)$, corresponding to the root systems (7) and (8). The vertical axes of each $\mathrm{SO}(4,2)$-tower are formed by the eigenvalues of the Cartan generator $\mathbf{\Delta}_3 = \mathsf{L}_{56}$, which adds a radial ladder operator to the variety. Each given floor of a $\mathrm{SO}(4,2)$-tower (Haenzel circle) is characterized by the main quantum number n. The horizontal bands (on the floors) correspond to various l-subshells (Haenzel rings), and the points are individual m-components (finite-dimensional representations of the group $\mathrm{SO}(4,2)$) defining the elements of the periodic table according to the Madelung rule. Rings containing elements with an odd sum of $n+l$ are indicated in yellow, respectively, rings with an even sum of $n+l$ are indicated in blue. Homologous elements are connected by vertical lines (Bailey-Thomsen-Bohr lines). Transitions between different quantum levels are indicated by directional lines (Haenzel lines). On the radial axes of the $\mathrm{SO}(4,2)$-towers are

hyperdodecahedron), it has no analogue among the five Platonic solids (regular three-dimensional polyhedra). The three-dimensional orthographic projection of the octaplex is a *cuboctahedron* (one of the 13 Archimedean solids). It should be noted that the four-dimensional space contains the largest number of different types of regular polyhedra, equal to 6. For all other n-dimensional spaces, for $n > 4$, this number is 3.

[3]The eigenvalue σ of the generator L_{78} corresponds to the fourth quantum number s.

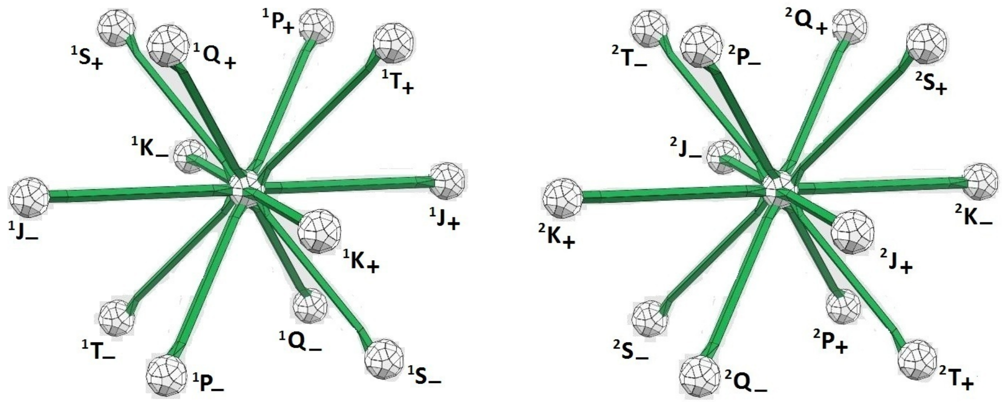

Figure 3. Root diagrams (cuboctahedra) of split bases (5) and (6) of the Lie algebra $\mathfrak{so}(4,4)$.

metals with quantum numbers $(n, l = 0, s = \pm 1/2)$, $n = 1, 2, \ldots$. On the radial axis of the $\mathrm{SO}(4,2)$-tower with the quantum number $s = -1/2$ (the left tower) are the alkali metals of group I (**Li**, **Na**, **K**, **Rb**, **Cs**, **Fr**, ...), respectively, on the right tower ($s = +1/2$), the radial axis is inhabited by group II alkaline earth metals (**Be**, **Mg**, **Ca**, **Sr**, **Ba**, **Ra**, ...). As we move away from the radial axes to the periphery of the Haenzel circles, the *metallicity* of the elements decreases and *nonmetallicity* increases, which corresponds to a movement from left to right along the period in the standard periodic table (metals $\rightarrow$ amphoteric elements $\rightarrow$ inert gases).

Further, Figure 5 shows two combined three-dimensional projections ($\mathrm{SO}(4,2)$-towers) of the weight diagram of the Lie algebra $\mathfrak{so}(4,4)$ of the rotation group $\mathrm{SO}(4,4)$ corresponding to the bases (5) and (6). It is easy to see that on each floor of the double $\mathrm{SO}(4,2)$-tower shown in Figure 5, the structure of (j,j)-Fock representations of the subgroup $\mathrm{SO}(4)$ (the integer part of the weight diagram of the subalgebra $\mathfrak{so}(4)$, see Figure 3 in [17]). The first floor ($n = 1$) contains the first period, consisting of hydrogen **H** and helium **He**. **H** and **He** form a double singlet. On the floor $n = 2$, the m-components (elements of the periodic table) form the $(1,1)$-diagram of the subalgebra $\mathfrak{so}(4)$, containing the second period: a double singlet (**Be**,**Li**) and two triplets (**B**,**C**,**N**), (**Ne**,**F**,**O**). For example, on the fourth $n = 4$ floor of the $\mathrm{SO}(4,2)$-tower, we come to the $(3,3)$-diagram of the subalgebra $\mathrm{SO}(4)$, which completes the filling of the 4th period with inert gas **Kr** (krypton), see Figure 6. $(3,3)$-diagram in Figure 6 also contains transition metals of the 5th period from yttrium **Y** to technetium **Tc** and from ruthenium **Ru** to cadmium **Cd**. Here, within the framework of the $(3,3)$-diagram of Figure 6, the entire lanthanide family **La**, **Ce**, ..., **Yb** is located. Thus, all the elements of the periodic table are grouped into a sequence of weight $\mathrm{SO}(4)$-diagrams of the subalgebra $\mathfrak{so}(4)$, located on eight floors of the weight diagram (double $\mathrm{SO}(4,2)$-tower, Figure 5) of the algebra $\mathfrak{so}(4,4)$.

On each floor of the left ($s = -1/2$) and right ($s = +1/2$) towers (see Figure 4) there are n^2 states, which corresponds to the dimension of representations of the subgroup $\mathrm{SO}(4)$, and on the floors of the combined tower (see Figure 5), we have implementations of weight diagrams of the algebra $\mathfrak{so}(4)$. Thus, the number of states (elements) on each floor

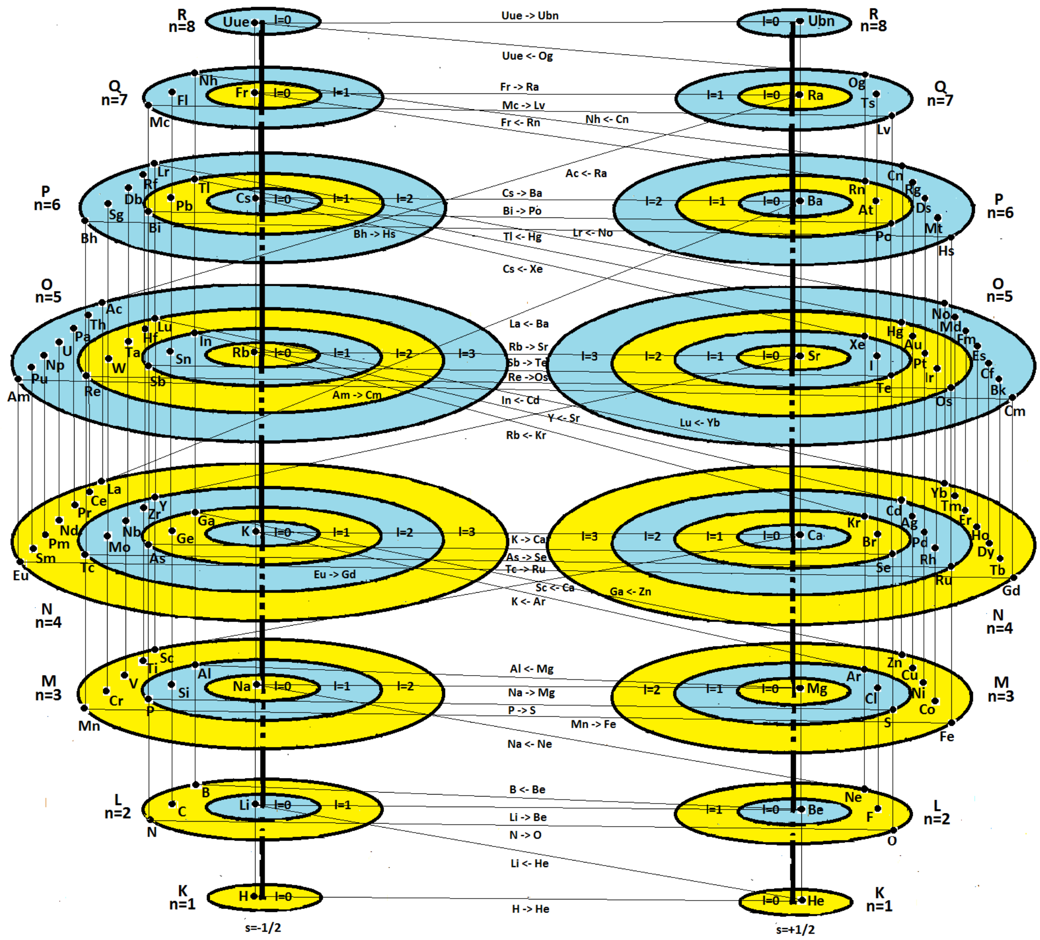

Figure 4. The periodic system of chemical elements in the split basis of the group algebra of the twofold covering $\mathbf{Spin}_+(4,2) \simeq \mathrm{SU}(2,2)$.

of a combined $\mathrm{SO}(4,2)$-tower is determined by the Rydberg sequence $2n^2$:

$$2 = 2 \cdot 1^2,\ 8 = 2 \cdot 2^2,\ 18 = 2 \cdot 3^2,\ 32 = 2 \cdot 4^2,\ \ldots, \tag{9}$$

which Sommerfeld called “cabalistic” in his book [20]. Then there is *period doubling* (except for the first one):

$$2,\ 8,\ 8,\ 18,\ 18,\ 32,\ 32,\ \ldots. \tag{10}$$

The first doubling (9) is sometimes called “horizontal” (or *spin*), the second (10) is “vertical”.

3. Spin and Period Doubling

P.-O. Löwdin notes in the article [21] that the lack of a theoretical explanation for the period doubling (as stated in [7, 22], which has been taking place so far) is equivalent to the lack of a theoretical understanding of the periodic table of chemical elements as a whole.

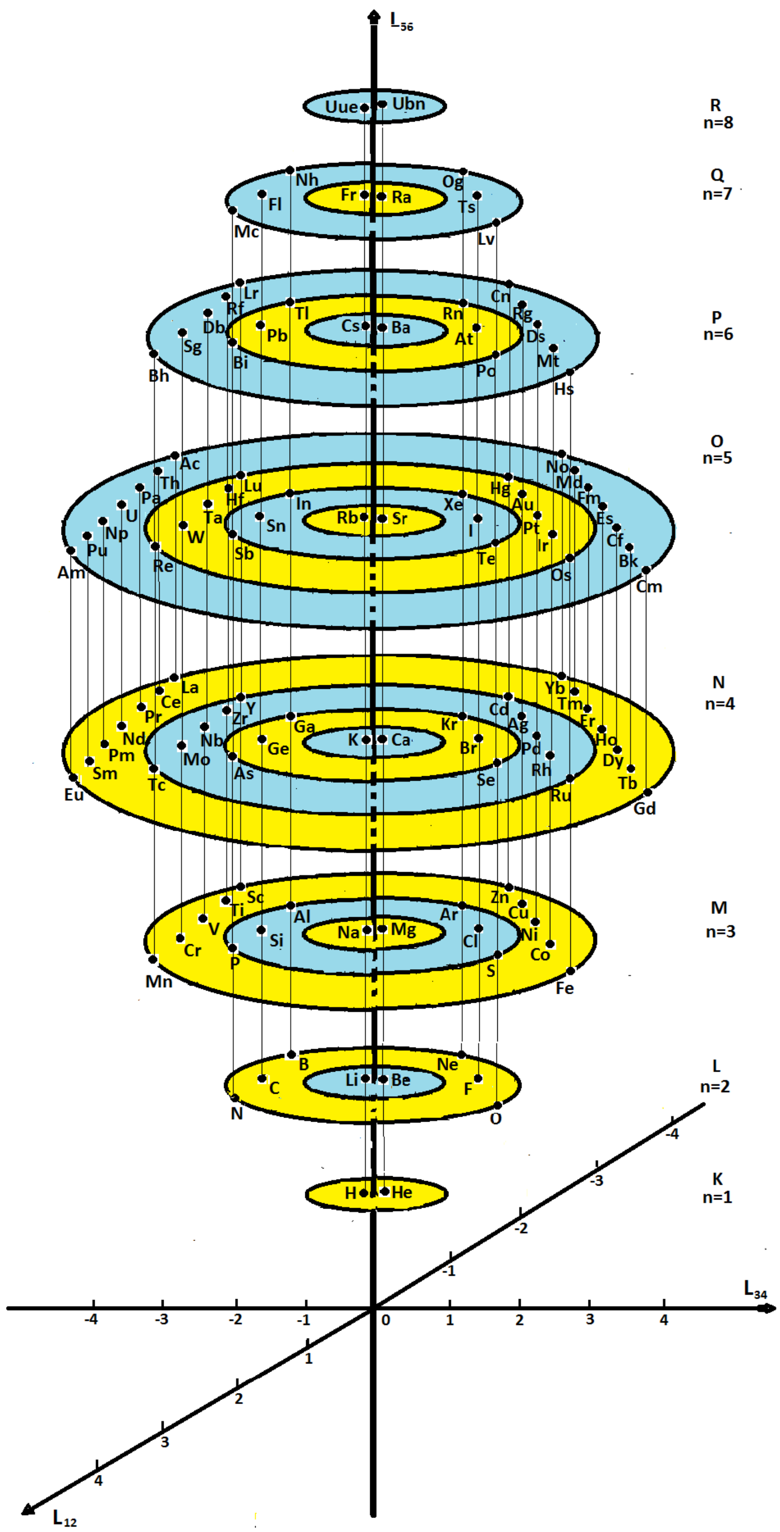

Figure 5. The periodic table of chemical elements in the form of a combined weight diagram of split bases (5) and (6) of the Lie algebra $\mathfrak{so}(4,4)$.

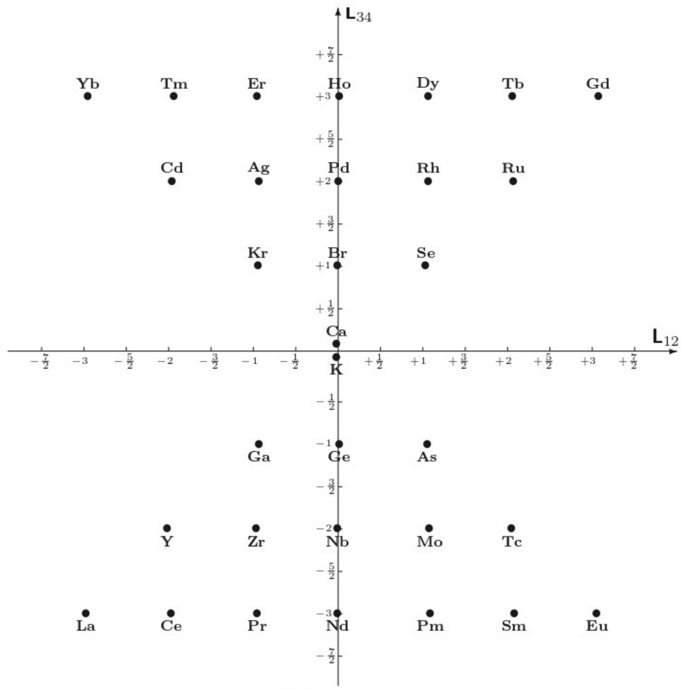

Figure 6. The level $n = 4$: the $(3,3)$-diagram of the subalgebra $\mathfrak{so}(4)$ contains the completion of the 4th period, transition metals of the 5th period **Y**, . . ., **Tc**, **Ru**, . . ., **Cd**, as well as the lanthanide family **La**, **Ce**, . . ., **Yb**.

Let's consider how the period doubling was described by Ostrovsky [24] and Fet[11][4]. Ostrovsky group [24]

$$G_O = \mathrm{O}(4,2) \otimes \mathrm{SU}(2)_S \otimes \mathrm{SU}(2)_T \tag{11}$$

and Fet's group [11]

$$G_F = \mathrm{SO}(4,2) \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(2)' \tag{12}$$

have a similar structure[5]. The subgroup $\mathrm{O}(4)\otimes\mathrm{SU}(2)_S\otimes\mathrm{SU}(2)_T$ in (11) contains the symmetry $\mathrm{O}(4)$, which leads to representations of dimension n^2. By extending the group $\mathrm{O}(4)$ to $\mathrm{O}(4) \otimes \mathrm{SU}(2)_S$, the dimensions of the representations are doubled to $2n^2$. The subscript S here indicates the physical origin of the group $\mathrm{SU}(2)$ from the electron spin $m_s = \pm 1/2$. Ostrovsky called this "horizontal" doubling of period lengths *spin doubling* (see (9)). Fet calls this *first doubling* $\mathrm{SO}(4,2)\otimes\mathrm{SU}(2)$, the representation of this group is $F_s^+ = \varphi_2 \otimes F^+$, where φ_2 is the unitary representation of the group $\mathrm{SU}(2)$ in the space $C(2)$, F^+ is an extension of the Fock representation F for the subgroup $\mathrm{SO}(4)$ to the conformal group $\mathrm{SO}(4,2)$.

[4]Earlier in these papers, Barut [12] tried to explain the period doubling by reducing representations of the conformal group $\mathrm{SO}(4,2)$ relative to the subgroup $\mathrm{SO}(3,2)$ (the anti-de Sitter group). O. Novaro [23] is also known for his attempt to explain the doubling of periods by distinguishing between two types of representations $(j,j,0)$ and $(j,0,j)$ of the Novaro-Berrondo group $G_{NB} = \mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(2)$.

[5]In earlier works, [25, 26] Fet interprets the period doubling by including the cyclic group $\mathbb{Z}_2$: $\mathrm{O}(4,2) \otimes \mathrm{SU}(2) \otimes \mathbb{Z}_2$.

This leads to *two copies* of the representations of the group $\mathrm{SO}(4,2)\otimes\mathbf{1}$, which are implemented in two different Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$. Next, Fet introduces three generators $\boldsymbol{\tau}_+$, $\boldsymbol{\tau}_-$ and $\boldsymbol{\tau}_3$ of the algebra $\mathfrak{su}(2)$, where $\boldsymbol{\tau}_3$ acts as the Cartan generator, which is permuted with all generators of the subgroup $\mathrm{SO}(4,2)\otimes\mathbf{1}$ and distinguishes states from both subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$, and the ladder operators $\boldsymbol{\tau}_\pm$ (Weyl generators) act as shift operators between $\mathcal{H}_+$ and $\mathcal{H}_-$. Figure 7 shows the Janet table in a pyramidal form, where the cells of the table (chemical elements) are accompanied by the quantum numbers of the Fet group (the first Fet basis). It can be seen from the figure that the horizontal (spin) doubling corresponds to the double splitting of the μ-components of the λ-multiplets. "Vertical" doubling

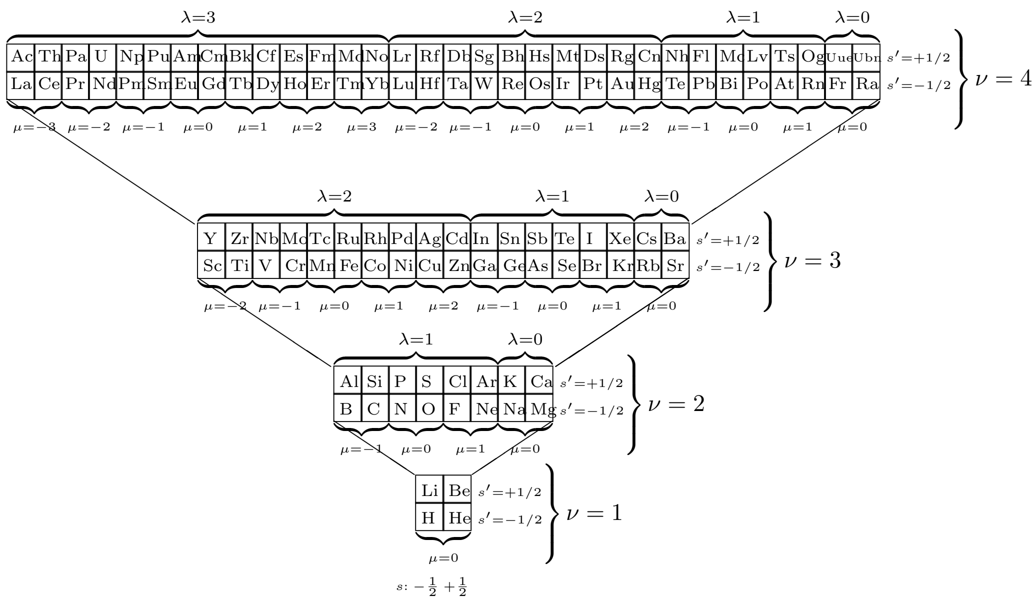

Figure 7. The Janet table in a pyramidal form.

of period lengths, known as the actual *period doubling* (see (10)) in the periodic table, was formulated by Ostrovsky in group-theoretic form by introducing a second group $\mathrm{SU}(2)$, denoted by $\mathrm{SU}(2)_T$ and formally analogous to the isospin group. Fet calls this *second doubling* $\mathrm{SO}(4,2)\otimes\mathrm{SU}(2)\otimes\mathrm{SU}(2)'$ (Fet group G_F), which is introduced similarly to the first doubling, i.e. the representation of this group has the form $F^+_{ss'}=\varphi'_2\otimes F^+_s$, where φ'_2 is the unitary representation of the group $\mathrm{SU}(2)'$ in the space $C(2)$. In turn, this leads to *two copies* of the representations of the group $\mathrm{SO}(4,2)\otimes\mathrm{SU}(2)\otimes\mathbf{1}$, which are implemented in two different Hilbert spaces $\mathcal{H}'_+$ and $\mathcal{H}'_-$. Further, Fet introduced three generators $\boldsymbol{\tau}'_+$, $\boldsymbol{\tau}'_-$ and $\boldsymbol{\tau}'_3$ of the algebra $\mathfrak{su}(2)'$, where $\boldsymbol{\tau}'_3$ acts as a Cartan generator that commutes with all generators of the subgroup $\mathrm{SO}(4,2)\otimes\mathrm{SU}(2)\otimes\mathbf{1}$ and distinguishes states from both subspaces $\mathcal{H}'_+$ and $\mathcal{H}'_-$, and the Weyl generators $\boldsymbol{\tau}'_\pm$ act as shift (ladder) operators between $\mathcal{H}'_+$ and $\mathcal{H}'_-$. In Figure 7, the vertical doubling given by *the fifth quantum number* s' corresponds to the double (vertical) splitting of the ν blocks (tiers of the Janet pyramid).

The main disadvantage of the Ostrovsky-Fet scheme of the group-theoretic description of period doubling is the artificial nature of the introduction of a fifth quantum number s', which has no real analogue, since all states (elements) of the periodic system are described by the four quantum numbers $(n, l, m.s)$. According to Fet [11], the set $(\nu, \lambda, \mu, s, s')$ of the five quantum numbers of the group G_F defines all the states of the periodic table, while the quantum number ν is equal to $\nu = 1/2(n + l + 1)$ for an odd value of the sum of $n + l$ and $\nu = 1/2(n + l)$ for an even value of $n + l$, which leads to a change in the Madelung rule.

Turning to the analysis of period doubling within the framework of $\mathrm{SO}(4, 4)$, first of all it should be noted that the group $\mathrm{SO}(4, 4)$ has a higher symmetry than tensor products[6] (11) and (12), which are subgroups of rotation groups of ten-dimensional spaces. Accordingly, their Lie algebras have fifth rank. As noted above in section 2, the fourth generator L_{78} of the Cartan subalgebra $\mathfrak{K} \subset \mathfrak{so}(4, 4)$, understood as a spin generator, commutes with all 15 generators of the subalgebra $\mathfrak{so}(4, 2)$, which leads to a double splitting of (5) and (6). The weight diagrams corresponding to the bases (5) and (6) are shown in Figure 4. In this case, the horizontal (spin) doubling of the Ostrovsky-Fet scheme is represented by two towers of representations: $s = -1/2$ (hydrogen line) and $s = 1/2$ (helium line). The "vertical" (actual) period doubling is given by the floors (Haenzel circles) K, L, M, N, O, P, Q, R of the corresponding $\mathrm{SO}(4, 2)$-towers (pyramid tiers representations of the conformal group, see Figure 7 in [17]). At the same time, unlike the Ostrovsky-Fet scheme, it does not require the introduction of an additional fifth quantum number.

Within the framework of the proposed scheme for describing the periodic table, the fourth generator L_{78} of the Cartan subalgebra of the algebra $\mathfrak{so}(4, 4)$ combines the functions of both generators $\boldsymbol{\tau}_3$ and $\boldsymbol{\tau}'_3$ of the Ostrovsky-Fet scheme by virtue of higher symmetry of the group $\mathrm{SO}(4, 4)$. Switching the generator L_{78} with all 15 generators of the subalgebra $\mathfrak{so}(4, 2)$ leads to splitting of the Cartan-Weyl basis of the algebra $\mathfrak{so}(4, 4)$ and, accordingly, to the first doubling (9). The weight diagram of the algebra $\mathfrak{so}(4, 4)$ is four-dimensional, the axis of the generator L_{78} is perpendicular to the axis of the radial generator L_{56}, as well as to the axes of the generators L_{12} and L_{34}, i.e. the axis of the generator L_{78} is perpendicular to the three-dimensional weight space, which is based on the axes of the generators L_{12}, L_{34}, L_{56}, which in turn leads to "vertical" (actual) period doubling (10). This eliminates the need to introduce a fifth quantum number.

Thus, *the actual period doubling (10) is the result of the action of the fourth generator* L_{78} *(spin generator) of the Cartan subalgebra* $\mathfrak{K}$ *of the Lie algebra* $\mathfrak{so}(4, 4)$. In this case, spin is the fourth degree of freedom, which has no analogue in the framework of the classical three-dimensional system. *Spin is an effect of the fourth dimension*[7].

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[6] The Lie algebra $\mathfrak{so}(4, 4)$ contains 28 independent generators (see section 2), while $\mathfrak{so}(4, 2) \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)'$ has 21 generator.

[7] That substantial three-dimensional image of an atom formed by means of recording equipment (mass spectrometers, electron and tunneling microscopes) is nothing more than the tip of an iceberg, the main part of which is immersed in the world of four (and maybe more) dimensions. Heisenberg said: "Atoms are not things".

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